A General Beurling-Helson-Lowdenslager Theorem on the Disk

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Abstract. The classical Beurling-Helson-Lowdenslager theorem characterizes the shift-invariant subspaces of the Hardy space H^2 and of the Lebesgue space L^2. In this paper, which is self-contained, we define a very general class of norms α and define spaces H^α and L^α. We then extend the Beurling-Helson-Lowdenslager invariant subspace theorem. The idea of the proof is new and quite simple; most of the details involve extending basic well-known $\|\|_p$-results for our more general norms.

1. Introduction

Why should we care about invariant subspaces? In finite dimensions all of the structure theorems for operators can be expressed in terms of invariant subspaces. For example the statement that every $n \times n$ complex matrix T is unitarily equivalent to an upper triangular matrix is equivalent to the existence of a chain $M_0 \subset M_1 \subset \cdots \subset M_n$ of T-invariant linear subspaces with $\dim M_k = k$ for $0 \leq k \leq n$. Since every upper triangular normal matrix is diagonal, the preceding result yields the spectral theorem. A matrix is similar to a single Jordan block if and only if its set of invariant subspaces is linearly ordered by inclusion, so the Jordan canonical form can be completely described in terms of invariant subspaces. In [**3**] L. Brickman and P. A. Fillmore describe the lattice of all invariant subspaces of an arbitrary matrix.

In infinite dimensions, where we consider closed subspaces and bounded operators, even the existence of one nontrivial invariant subspace remains an open problem for Hilbert spaces. If T is a normal operator with a $*$-cyclic vector, then, by the spectral theorem, T is unitarily equivalent to the multiplication operator M_z on $L^2(\sigma(T), \mu)$, i.e.,

$$(M_z f)(z) = zf(z),$$

2000 *Mathematics Subject Classification.* Primary 46L52, 30H10; Secondary 47A15.

Key words and phrases. Hardy space, $\|\|_1$-dominating normalized gauge norm, dual space, invariant subspace, Beurling-Helson-Lowdenslager theorem.

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where μ is a probability Borel measure on the spectrum $\sigma(T)$ of T. In this case von Neumann proved that if a subspace W that is invariant for M_z and for $M_z^* = M_{\bar{z}}$, then the projection P onto W is in the commutant of M_z, which is the maximal abelian algebra $\{M_\varphi : \varphi \in L^\infty(\mu)\}$. Hence there is a Borel subset E of $\sigma(T)$ such that $P = M_{\chi_E}$, which implies $W = \chi_E L^2(\mu)$. It follows that if T is a *reductive* normal operator, i.e., every invariant subspace for T is invariant for T^*, then all invariant subspaces of T have the form $\chi_E L^2(\mu)$. In [**9**] D. Sarason characterized the (M_z, μ) that are reductive; in particular, when $T = M_z$ is unitary (i.e., $\sigma(T) \subset \mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$), then M_z is reductive if and only if Haar measure m on $\mathbb{T}$ is not absolutely continuous with respect to μ. When $\sigma(T) = \mathbb{T}$ and $\mu = m$ is Haar measure on $\mathbb{T}$, then M_z on L^2 is the bilateral shift operator.

If T is the restriction of a normal operator to an invariant subspace with a cyclic vector e, then there is a probability space $L^2(\sigma(T), \mu)$ such that T is unitarily equivalent to M_z restricted to $P^2(\mu)$, the closure of the polynomials in $L^2(\mu)$, and where e corresponds to the constant function $1 \in P^2(\mu)$. If $\sigma(T) = \mathbb{T}$ and $\mu = m$, then $P^2(\mu)$ is the classical Hardy space H^2 and M_z is the unilateral shift operator.

In infinite dimensions the first important characterization of all the invariant subspaces of a non-normal operator, the unilateral shift, was due to A. Beurling [**1**] in 1949. His result was extended by H. Helson and D. Lowdenslager [**7**] to the bilateral shift operator, which is a non-reductive unitary operator.

In this paper, suppose $\mathbb{D}$ is the unit disk in the complex plane $\mathbb{C}$, and m is Haar measure (i.e., normalized arc length) on the unit circle $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. We let $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, respectively, denote the sets of real numbers, integers, and positive integers. Since $\{z^n : n \in \mathbb{Z}\}$ is an orthonormal basis for L^2, we see that M_z is a *bilateral shift* operator. The subspace H^2 which is the closed span of $\{z^n : n \geq 0\}$ is invariant for M_z and the restriction of M_z to H^2 is a *unilateral shift* operator. A closed linear subspace W of L^2 is *doubly invariant* if $zW \subseteq W$ and $\bar{z}W \subseteq W$. Since $\bar{z}z = 1$ on $\mathbb{T}$, W is doubly invariant if and only if $zW = W$. Since the set of polynomials in z and $\bar{z}$ is weak*-dense in $L^\infty = L^\infty(\mathbb{T})$, and since the weak* topology on L^∞ coincides with the weak operator topology on L^∞ (acting as multiplication operators on L^2), W is doubly invariant if and only if $L^\infty \cdot W \subseteq W$. A subspace W is *simply invariant* if $zW \subsetneqq W$, which means $H^\infty \cdot W \subseteq W$, but $\bar{z}W \nsubseteq W$.

We state the classical Beurling-Helson-Lowdenslager theorem for a closed subspace W of L^2. A very short elegant proof is given in [**8**]. We give a short proof to make this paper self-contained.

THEOREM 1.1. ***(Beurling-Helson-Lowdenslager)*** *Suppose W is a closed linear subspace of L^2 and $zW \subset W$. Then*

(1) *if W is doubly invariant, then $W = \chi_E L^2$ for some Borel subset E of $\mathbb{T}$;*
(2) *if W is simply invariant, then $W = \varphi H^2$ for some $\varphi \in L^\infty$ with $|\varphi| = 1$ a.e. (m);*
(3) *if $0 \neq W \subseteq H^2$, then $W = \varphi H^2$ with φ an inner function (i.e., $\varphi \in H^\infty$ and $|\varphi| = 1$ a.e. (m)).*

PROOF. 1. This follows from von Neumann's result discussed above.

2. If W is simply invariant, then $M_z|W$ is a nonunitary isometry, which, by the Halmos-Wold-Kolmogorov decomposition must be a direct sum of at least one unilateral shift and an

isometry. Thus $W = W_1 \oplus W_2$ and there is a unit vector $\varphi \in W_1$ with $\{z^n\varphi : n \geq 0\}$ an orthonormal basis for W_1. Since $\varphi \perp z^n\varphi$ for all $n \geq 1$, we have

$$\int_{\mathbb{T}} |\varphi|^2 z^n dm = 0$$

for all $n \geq 1$, and taking conjugates, we also get the same result for all $n \leq -1$, which implies that $|\varphi(z)|^2 = \sum_{n=-\infty}^{\infty} c_n z^n = c_0$. Since φ is a unit vector, we have $|\varphi|^2 = 1$ a.e. (m). Hence,

$$W_1 = \varphi \cdot \overline{sp}\left(\{z^n : n \geq 0\}\right) = \varphi H^2.$$

If g is a unit vector in W_2, then we have $z^n\varphi \perp g$ and $\varphi \perp z^n g$ for $n \geq 0$, which yields

$$\int_{\mathbb{T}} z^n \varphi \bar{g} dm = 0$$

for all $n \in \mathbb{Z}$. It follows from the definition of Fourier coefficients of φg that $|\varphi\bar{g}| = 0$, which implies $|g| = |\varphi\bar{g}| = 0$. Hence $W = W_1 = \varphi H^2$.

3. It is clear that no nonzero subspace $W \supseteq zW$ of H^2 can have the form $\chi_E L^2$, so the only possibility is the situation in part (2), which means $W = \varphi H^2$. Also, $\varphi \in \varphi H^2 = W \subset H^2$, which implies $\varphi \in H^2$ is inner. $\square$

These results are also true when $\|\|_2$ is replaced with $\|\|_p$ for $1 \leq p \leq \infty$, with the additional assumption that W is weak*-closed when $p = \infty$ (see [**10**], [**11**], [**12**]). Many of the proofs for the $\|\|_p$ case use the L^2 result and take cases when $p \leq 2$ and $2 < p$. In [**4**] and [**5**], the author proved version of part (3) for a large class of norms, called *rotationally invariant* norms. In these more general setting, the cases $p \leq 2$ and $2 < p$ have no analogue.

In this paper we extend the Beurling-Helson-Lowdenslager theorem to an even larger class of norms, the *continuous* $\|\|_1$*-dominating normalized gauge norms*, with a proof that is simple even in the L^p case. For each such norm α we define a Banach space L^α and a Hardy space H^α with

$$L^\infty \subset L^\alpha \subset L^1 \text{ and } H^\infty \subset H^\alpha \subset H^1.$$

In this new setting, we prove the following Beurling-Helson-Lowdenslager theorem, which is the main result of this paper.

Theorem 3.6. Suppose α is a continuous $\|\|_1$-dominating normalized gauge norm and W is a closed subspace of L^α. Then $zW \subseteq W$ if and only if either $W = \phi H^\alpha$ for some unimodular function ϕ or $W = \chi_E L^\alpha$ for some Borel set $E \subset \mathbb{T}$. If $0 \neq W \subset H^\alpha$, then $W = \varphi H^\alpha$ for some inner function φ.

To prove Theorem 3.6. we need the following technical theorem in Section 3.

Theorem 3.4. Suppose α is a continuous $\|\|_1$-dominating normalized gauge norm. Let W be an α-closed linear subspace of L^α, and M be a weak*-closed linear subspace of L^∞ such that $zM \subseteq M$ and $zW \subseteq W$. Then

(1) $M = [M]^{-\alpha} \cap L^\infty$;

(2) $W \cap M$ is weak*-closed in L^∞;
(3) $W = [W \cap L^\infty]^{-\alpha}$.

This gives us a quick route from the $\| \|_2$-version of invariant subspace structure to the (weak*-closed) $\| \|_\infty$-version of invariant subspace structure, and then to the α-version of invariant subspace structure.

2. Preliminaries

DEFINITION 2.1. *A norm α on L^∞ is called a* $\| \|_1$-dominating normalized gauge norm *if*
(1) $\alpha(1) = 1$,
(2) $\alpha(|f|) = \alpha(f)$ *for every* $f \in L^\infty$,
(3) $\alpha(f) \geq \|f\|_1$ *for every* $f \in L^\infty$.

We say that a $\| \|_1$-dominating normalized gauge norm is *continuous* if

$$\lim_{m(E)\to 0^+} \alpha(\chi_E) = 0.$$

Although a $\| \|_1$-dominating normalized gauge norm α is defined only on L^∞, we can define α for all measurable functions f on $\mathbb{T}$ by

$$\alpha(f) = \sup\{\alpha(s) : s \text{ is a simple function}, 0 \leq s \leq |f|\}.$$

It is clear that $\alpha(f) = \alpha(|f|)$ still holds.

We define

$$\mathcal{L}^\alpha = \{f : \alpha(f) < \infty\},$$

and define L^α to be the α-closure of L^∞ in $\mathcal{L}^\alpha$.

LEMMA 2.2. *Suppose $f, g : \mathbb{T} \to \mathbb{C}$ are measurable. Let α be a $\| \|_1$-dominating normalized gauge norm. Then the following statements are true:*
(1) $|f| \leq |g| \Longrightarrow \alpha(f) \leq \alpha(g)$;
(2) $\alpha(fg) \leq \alpha(f)\|g\|_\infty$;
(3) $\alpha(g) \leq \|g\|_\infty$;
(4) $L^\infty \subset L^\alpha \subset \mathcal{L}^\alpha \subset L^1$;
(5) *If α is continuous, $0 \leq f_1 \leq f_2 \leq \cdots$ and $f_n \to f$ a.e. (m), then $\alpha(f_n) \to \alpha(f)$;*
(6) *If α is continuous, then $\mathcal{L}^\alpha$ and L^a are both Banach spaces.*

PROOF. 1. It is clear that if $|u| = 1$ a.e. (m), then

$$\alpha(uf) = \alpha(|uf|) = \alpha(|f|) = \alpha(f).$$

If $|f| \leq |g|$, then there is a measurable h with $|h| \leq 1$ a.e. (m) such that $f = hg$. Then there are measurable functions u_1 and u_2 with $|u_1| = |u_2| = 1$ a.e. (m) with $h = (u_1 + u_2)/2$. Hence

$$\alpha(f) = \alpha((u_1 g + u_2 g)/2) \leq \frac{1}{2}[\alpha(u_1 g) + \alpha(u_2 g)] = \alpha(g).$$

2. This follows from part (1) and the fact that $|fg| \leq |f| \|g\|_\infty$ a.e. (m).
3. This follows from part (2) with $f = 1$.

4. Since $\|s\|_1 \leq \alpha(s)$ whenever $s \in L^\infty$, it follows for any measurable f that

$$\alpha(f) \geq \sup\{\|s\|_1 : s \text{ is a simple function}, 0 \leq s \leq |f|\} = \|f\|_1 .$$

This implies $\mathcal{L}^\alpha \subset L^1$. The inclusions $L^\infty \subset L^\alpha \subset \mathcal{L}^\alpha$ follow from the definition of L^α.

5. Suppose $0 \leq s \leq f$ and $0 \leq t < 1$. Write $s = \sum_{1\leq k\leq m} a_k \chi_{E_k}$ with $0 < a_k$ for $1 \leq k \leq m$ and $\{E_1, \ldots, E_m\}$ disjoint. If we let $E_{k,n} = \{\omega \in E_k : ta_k < f_n(\omega)\}$, we see that

$$E_{k,1} \subset E_{k,2} \subset \cdots \quad \text{and} \quad \cup_{1\leq n<\infty} E_{k,n} = E_k.$$

Since α is continuous,

$$\alpha\left(\chi_{E_k} - \chi_{E_{k,n}}\right) = \alpha\left(\chi_{E_k \setminus E_{k,n}}\right) \to 0.$$

Hence

$$t\alpha(s) = \lim_{n\to\infty} \alpha\left(\sum_{k=1}^m ta_k \chi_{E_{k,n}}\right) \leq \lim_{n\to\infty} \alpha(f_n).$$

Since t was arbitrary, for every simple function s with $0 \leq s \leq f$, we have

$$\alpha(s) \leq \lim_{n\to\infty} \alpha(f_n).$$

By the definition of $\alpha(f)$, we see that $\alpha(f) \leq \lim_{n\to\infty} \alpha(f_n)$, and $\alpha(f_n) \leq \alpha(f)$ for each $n \geq 1$ follows from part (1), which implies $\lim_{n\to\infty} \alpha(f_n) \leq \alpha(f)$. This completes the proof.

6. It follows from the definition of $\mathcal{L}^\alpha$ that $\mathcal{L}^\alpha$ is a normed space with respect to α. To prove the completeness, suppose $\{f_n\}$ is a sequence in $\mathcal{L}^\alpha$ with $\sum_{n=1}^\infty \alpha(f_n) < \infty$. Since $\|\|_1 \leq \alpha()$, we know that $g = \sum_{n=1}^\infty |f_n| \in L^1$ and $f = \sum_{n=1}^\infty f_n$ converges a.e. (m). It follows from part (5) that

$$\alpha(g) = \lim_{N\to\infty} \alpha\left(\sum_{n=1}^N |f_n|\right) \leq \lim_{N\to\infty} \sum_{n=1}^N \alpha(|f_n|) = \sum_{n=1}^\infty \alpha(f_n) < \infty.$$

Since $|f| \leq g$, from part (1) we know $\alpha(f) < \infty$. Thus $f \in \mathcal{L}^\alpha$. Also, by part (1) and part (5), we have for each $N \geq 1$,

$$\alpha\left(f - \sum_{n=1}^N f_n\right) \leq \alpha\left(\sum_{n=N+1}^\infty |f_n|\right) \leq \sum_{n=N+1}^\infty \alpha(f_n) = \sum_{n=1}^\infty \alpha(f_n) - \sum_{n=1}^N \alpha(f_n).$$

Hence

$$\lim_{N\to\infty} \alpha\left(f - \sum_{n=1}^N f_n\right) = 0.$$

Since every absolutely convergent series in $\mathcal{L}^\alpha$ is convergent in $\mathcal{L}^\alpha$, we know $\mathcal{L}^\alpha$ is complete. $\square$

We let $\mathcal{N}$ denote the set of all $\|\|_1$-dominating normalized gauge norms, and we let $\mathcal{N}_c$ denote the set of continuous ones. We can give $\mathcal{N}$ the topology of pointwise convergence.

Lemma 2.3. *The sets $\mathcal{N}$ and $\mathcal{N}_c$ are convex, and the set $\mathcal{N}$ is compact in the topology of pointwise convergence.*

Proof. It directly follows from the definitions that $\mathcal{N}$ and $\mathcal{N}_c$ are convex. Suppose $\{\alpha_\lambda\}$ is a net in $\mathcal{N}$ and choose a subnet $\{\alpha_{\lambda_\kappa}\}$ that is an ultranet. Then, for every $f \in L^\infty$, we have that $\{\alpha_{\lambda_\kappa}(f)\}$ is an ultranet in the compact set $[\|f\|_1, \|f\|_\infty]$. Thus

$$\alpha(f) = \lim_k \alpha_{\lambda_k}(f)$$

exists. It follows directly from the definition of $\mathcal{N}$ that $\alpha \in \mathcal{N}$. Hence $\mathcal{N}$ is compact. $\square$

Example 2.4. *There are many interesting examples of continuous $\|\|_1$-dominating normalized gauge norms other than the usual $\|\|_p$ norms ($1 \leq p < \infty$).*

1. *If $1 \leq p_n < \infty$ for $n \geq 1$, then $\alpha = \sum_{n=1}^\infty \frac{1}{2^n} \|\|_{p_n} \in \mathcal{N}_c$. Moreover if $p_n \to \infty$, then α is not equivalent to any $\|\|_p$.*
2. *The Lorentz, Marcinkiewicz and Orlicz norms are important examples (e.g., see [**2**]).*

Definition 2.5. *Let α be a $\|\|_1$-dominating normalized gauge norm. We define the* dual norm *$\alpha' : L^\infty \to [0, \infty]$ by*

$$\begin{aligned}\alpha'(f) &= \sup\left\{\left|\int_{\mathbb{T}} fh dm\right| : h \in L^\infty, \alpha(h) \leq 1\right\}\\ &= \sup\left\{\int_{\mathbb{T}} |fh|\, dm : h \in L^\infty, \alpha(h) \leq 1\right\}.\end{aligned}$$

Lemma 2.6. *Let α be a $\|\|_1$-dominating normalized gauge norm. Then the dual norm α' is also a $\|\|_1$-dominating normalized gauge norm.*

Proof. Suppose $f \in L^\infty$. If $h \in L^\infty$ with $\alpha(h) \leq 1$, then

$$\int_{\mathbb{T}} |fh| dm \leq \|f\|_\infty \|h\|_1 \leq \|f\|_\infty \alpha(h) \leq \|f\|_\infty,$$

thus $\alpha'(f) \leq \|f\|_\infty$. On the other hand, since $\alpha(1) = 1$, we have

$$\alpha'(f) \geq \int_{\mathbb{T}} |f| 1 dm = \|f\|_1,$$

it follows that α' is a norm with $\alpha'(|f|) = \alpha'(f)$ for every $f \in L^\infty$. Since $\|1\|_1 \leq \alpha'(1) \leq \|1\|_\infty$, we see that $\alpha'(1) = 1$. $\square$

Now we are ready to describe the dual space of L^α, when α is a continuous $\|\|_1$-dominating normalized gauge norm.

Proposition 2.7. *Let α be a continuous $\|\|_1$-dominating normalized gauge norm and let α' be the dual norm of α as in Definition 2.5. Then $(L^\alpha)^\sharp = \mathcal{L}^{\alpha'}$, i.e., for every $\phi \in (L^\alpha)^\sharp$, there is an $h \in \mathcal{L}^{\alpha'}$ such that $\|\phi\| = \alpha'(h)$ and*

$$\phi(f) = \int_{\mathbb{T}} fh dm$$

for all $f \in L^\alpha$.

Proof. If $\{E_n\}$ is a countable collection of disjoint Borel subsets of $\mathbb{T}$, it follows that

$$\lim_{N\to\infty} m\left(\cup_{n=N+1}^{\infty} E_n\right) = 0,$$

and the continuity of α implies

$$\lim_{N\to\infty} \left|\phi\left(\sum_{n=N+1}^{\infty} \chi_{E_n}\right)\right| \leq \lim_{N\to\infty} \|\phi\| \, \alpha\left(\sum_{n=N+1}^{\infty} \chi_{E_n}\right) = 0.$$

Hence

$$\phi\left(\sum_{n=1}^{\infty} \chi_{E_n}\right) = \sum_{n=1}^{\infty} \phi\left(\chi_{E_n}\right).$$

It follows that the restriction of ϕ to L^∞ is weak*-continuous, which implies there is an $h \in L^1$ such that, for every $f \in L^\infty$,

$$\phi(f) = \int_{\mathbb{T}} f h dm.$$

Since L^∞ is dense in L^α, it follows that

$$\phi(f) = \int_{\mathbb{T}} f h dm$$

holds for all $f \in L^\alpha$. Moreover, the definitions of α' and $\|\phi\|$ imply that $\alpha'(h) = \|\phi\|$. □

Corollary 2.8. *Let α be a continuous $\|\|_1$-dominating normalized gauge norm and let α' be the dual norm of α as in Definition 2.5. Then $\mathcal{L}^{\alpha'}$ is a Banach space.*

Proof. It follows from Theorem 2.7 that $\mathcal{L}^{\alpha'}$ is the dual space of L^α, thus $\mathcal{L}^{\alpha'}$ is a Banach space. □

We let $\mathbb{B} = \{f \in L^\infty : \|f\|_\infty \leq 1\}$ denote the closed unit ball in L^∞.

Lemma 2.9. *Let α be a continuous $\|\|_1$-dominating normalized gauge norm. Then*

(1) *The α-topology and the $\|\|_2$-topology coincide on $\mathbb{B}$.*
(2) *$\mathbb{B} = \{f \in L^\infty : \|f\|_\infty \leq 1\}$ is α-closed.*

Proof. 1. Since α is $\|\|_1$-dominating, α-convergence implies $\|\|_1$-convergence, which implies convergence in measure. Suppose $\{f_n\}$ is a sequence in $\mathbb{B}$ and $f \in \mathbb{B}$ with $f_n \to f$ in measure and $\varepsilon > 0$. If $E_n = \left\{z \in \mathbb{T} : |f(z) - f_n(z)| \geq \frac{\varepsilon}{2}\right\}$, then $\lim_{n\to\infty} m(E_n) = 0$. Since α is continuous, we have $\lim_{n\to\infty} \alpha(\chi_{E_n}) = 0$, which implies that

$$\begin{aligned}\alpha\left(f_n-f\right) &= \alpha((f-f_n)\chi_{E_n}+(f-f_n)\chi_{\mathbb{T}\backslash E_n}) \\ &\leq \alpha((f-f_n)\chi_{E_n})+\alpha((f-f_n)\chi_{\mathbb{T}\backslash E_n}) \\ &< \alpha(|f-f_n|\chi_{E_n})+\frac{\varepsilon}{2} \\ &\leq \|f-f_n\|_\infty\alpha(\chi_{E_n})+\frac{\varepsilon}{2} && \text{(by Part (2) of Lemma 2.2)} \\ &\leq 2\alpha(\chi_{E_n})+\frac{\varepsilon}{2}. && (f, f_n\in\mathbb{B}\text{ for all } n\geq 1)\end{aligned}$$

Hence $\alpha\left(f_n-f\right)\to 0$ as $n\to\infty$. Therefore α-convergence is equivalent to convergence in measure on $\mathbb{B}$. Since α was arbitrary, the same holds for $\|\|_2$-convergence.

2. Suppose $\{g_n\}$ is a sequence in $\mathbb{B}$ and $g\in L^\infty$ such that $\alpha(g_n-g)\to 0$. Since $\|\|_1\leq\alpha()$, it follows that $\|g_n-g\|_1\to 0$, which implies that $g_n\to g$ in measure. Then there is a subsequence $\{g_{n_k}\}$ such that $g_{n_k}\to g$ a.e. (m). Since $g_{n_k}\in\mathbb{B}$ for each $k\in\mathbb{N}$, we can conclude that $|g|\leq 1$, and hence $g\in\mathbb{B}$. This completes the proof. $\square$

3. The Main Result

In this section we prove our generalization of the classical Beurling-Helson-Lowdenslager theorem. Suppose α is a continuous $\|\|_1$-dominating normalized gauge norm. We define H^α to be the α-closure of H^∞, i.e.,

$$H^\alpha=\left[H^\infty\right]^{-\alpha}.$$

Since the polynomials in the unit ball $\mathbb{B}$ of H^∞ are $\|\|_2$-dense in $\mathbb{B}$ (consider the Cesaro means of the sequence of partial sums of the power series), we know from Lemma 2.9 that H^α is the α-closure of the set of polynomials.

The following lemma extends the classical characterization of $H^p=H^1\cap L^p$ to a more general setting.

Lemma 3.1. *Let α be a continuous $\|\|_1$-dominating normalized gauge norm. Then*

$$H^\alpha=H^1\cap L^\alpha.$$

Proof. Since α is $\|\|_1$-dominating, α-convergence implies $\|\|_1$-convergence, thus $H^\alpha=[H^\infty]^{-\alpha}\subset[H^\infty]^{-\|\|_1}=H^1$. Also, $H^\alpha=[H^\infty]^{-\alpha}\subset[L^\infty]^{-\alpha}=L^\alpha$, thus $H^\alpha\subset H^1\cap L^\alpha$. Now we suppose $0\neq f\in H^1\cap L^\alpha$ and $\varphi\in(L^\alpha)^\#$ with $\varphi|H^\alpha=0$. It follows from Proposition 2.7 that there is an $h\in\mathcal{L}^{\alpha'}$ such that $\varphi\left(v\right)=\int_{\mathbb{T}}vhdm$ for all $f\in L^\alpha$. From part (4) of Lemma 2.2 and Lemma 2.6, we see $h\in\mathcal{L}^{\alpha'}\subset L^1$, so we can write $h\left(z\right)=\sum_{n=-\infty}^{\infty}c_nz^n$. Since $\varphi|H^\alpha=0$, we have

$$c_{-n}=\int_{\mathbb{T}}hz^ndm=\varphi(z^n)=0$$

for all $n \geq 0$. Thus h is analytic and $h(0) = 0$. The fact that $h \in \mathcal{L}^{\alpha'}$ and $f \in L^{\alpha} \cap H^{1}$ imply that fh is analytic and $fh \in L^{1}$, which means $fh \in H^{1}$. Hence

$$\varphi(f) = \int_{\mathbb{T}} fh dm = f(0)\, h(0) = 0.$$

Since $\varphi|H^{\alpha} = 0$ and $f \neq 0$, it follows from the Hahn Banach theorem that $f \in H^{\alpha}$, which implies $H^{1} \cap L^{\alpha} \subset H^{\alpha}$. Therefore $H^{1} \cap L^{\alpha} = H^{\alpha}$. $\square$

A key ingredient is based on the following result that uses the Herglotz kernel [**6**].

Lemma 3.2. *$\{|g| : 0 \neq g \in H^{1}\} = \{\varphi \in L^{1} : \varphi \geq 0$ and $\log \varphi \in L^{1}\}$. In fact, if $\varphi \geq 0$ and $\varphi,\ \log \varphi \in L^{1}$, then*

$$g(z) = \exp \int_{\mathbb{T}} \frac{w+z}{w-z} \log \varphi(w)\, dm(w)$$

defines an outer function g on $\mathbb{D}$ and $|g| = \varphi$ on $\mathbb{T}$.

Combining Lemma 3.1 and Lemma 3.2, we obtain the following factorization theorem.

Proposition 3.3. *Let α be a continuous $\| \|_{1}$-dominating normalized gauge norm. If $k \in L^{\infty}$ and $k^{-1} \in L^{\alpha}$, then there is a unimodular function $w \in L^{\infty}$ and an outer function $h \in H^{\infty}$ such that $k = wh$ and $h^{-1} \in H^{\alpha}$.*

Proof. Recall that an outer function is uniquely determined by its absolute boundary values, which are necessarily absolutely log integrable. Suppose $k \in L^{\infty}$ with $k^{-1} \in L^{\alpha}$. Observe that on the unit circle,

$$-|k| \leq -\log|k| = \log|k^{-1}| \leq |k^{-1}|,$$

it follows from $k \in L^{\infty}$ and $k^{-1} \in L^{\alpha} \subset L^{1}$ that

$$-\infty < -\int_{\mathbb{T}} |k| dm \leq \int_{\mathbb{T}} \log|k^{-1}| dm \leq \int_{\mathbb{T}} |k^{-1}| dm < \infty,$$

and hence $|k^{-1}|$ is log integrable. Then by Lemma 3.2, there is an outer function $g \in H^{1}$ such that $|g| = |k^{-1}|$ on $\mathbb{T}$. If we let $h = g^{-1}$, then let $w = kg$. Since g is outer, $h = g^{-1}$ is analytic on $\mathbb{D}$. Also, on the unit circle $\mathbb{T}$, $|h| = |g^{-1}| = |k| \in L^{\infty}$, so $h \in H^{\infty}$. Moreover, $|g| = |k^{-1}|$ implies $w = kg$ is unimodular. Hence $k = wh$ where w is unimodular and $h \in H^{\infty}$. Furthermore, it follows from Lemma 3.1 that $h^{-1} = g = wk^{-1} \in L^{\alpha} \cap H^{1} = H^{\alpha}$. $\square$

Before we state our main results, we need the following lemmas.

Lemma 3.4. *Suppose X is a Banach space and M is a closed linear subspace of X. Then M is weakly closed.*

Lemma 3.5. *Let α be a continuous rotationally symmetric norm . If M is a closed subspace of H^{α} invariant under M_{z}, which means $zM \subset M$, then $H^{\infty}M \subset M$.*

PROOF. Let $\mathcal{P}_+ = \{e_n : n \in \mathbb{N}\}$ denote the class of all polynomials in H^∞, where $e_n(z) = z^n$ for all z in the unit circle $\mathbb{T}$. Since $zM \subset M$, we see $P(z)M \subset M$ for any polynomial $P \in \mathcal{P}_+$. To complete the proof, it suffices to show that $fh \in M$ for every $h \in M$ and every $f \in H^\infty$. Now we assume that u is a nonzero element in $\mathcal{L}^{\alpha'}$, then it follows from Proposition 2.7 that $hu \in M\mathcal{L}^{\alpha'} \subset L^\alpha(L^\alpha)^\sharp \subset L^1$. Since $f \in H^\infty$, we obtain $\hat{f}(n) = \int_{\mathbb{T}} f(z)z^{-n}dm(z) = 0$ for all $n < 0$, which implies that the partial sums $S_n(f) = \sum_{k=-n}^{n} \hat{f}(n)e_n = \sum_{k=0}^{n} \hat{f}(n)e_n \in \mathcal{P}_+$ for all $n < 0$. Hence the Cesaro means

$$\sigma_n(f) = \frac{S_0(f) + S_1(f) + \ldots + S_n(f)}{n+1} \in \mathcal{P}_+.$$

Moreover, we know that $\sigma_n(f) \to f$ in the weak* topology. Since $hu \in L^1$, we have

$$\int_{\mathbb{T}} \sigma_n(f)hudm \to \int_{\mathbb{T}} fhudm.$$

Observe that $\sigma_n(f)h \in \mathcal{P}_+M \subset M \subset L^\alpha(\mathbb{T})$ and $u \in \mathcal{L}^{\alpha'}(\mathbb{T})$, it follows that $\sigma_n(f)h \to fh$ weakly. Since M is a closed subspace of $H^\alpha(\mathbb{T})$, by Lemma 3.4, we see that M is weakly closed, which means $fh \in M$. This completes the proof. □

The following Lemma is the Krein-Smulian theorem.

LEMMA 3.6. *Let X be a Banach space. A convex set in $X^\sharp$ is weak* closed if and only if its intersection with $\{\phi \in X^\sharp : \|\phi\| \leq 1\}$ is weak* closed.*

The following theorem gives us a most general version of invariant subspace structure.

THEOREM 3.7. *Suppose α is a continuous $\|\|_1$-dominating normalized gauge norm. Let W be an α-closed linear subspace of L^α, and M be a weak*-closed linear subspace of L^∞ such that $zM \subseteq M$ and $zW \subseteq W$. Then*
(1) $M = [M]^{-\alpha} \cap L^\infty$;
(2) *$W \cap M$ is weak*-closed in L^∞;*
(3) $W = [W \cap L^\infty]^{-\alpha}$.

PROOF. 1. It is clear that $M \subset [M]^{-\alpha} \cap L^\infty$. Assume, via contradiction, that $w \in [M]^{-\alpha} \cap L^\infty$ and $w \notin M$. Since M is weak*-closed, there is an $F \in L^1$ such that $\int_{\mathbb{T}} Fwdm \neq 0$ but $\int gFdm = 0$ for every $g \in M$. Since $k = \frac{1}{|F|+1} \in L^\infty$ and $k^{-1} \in L^1$, it follows from Lemma 3.3 that there is an $h \in H^\infty$, $1/h \in H^1$ and a unimodular function u such that $k = uh$. Choose a sequence $\{h_n\}$ in H^∞ such that $\|h_n - 1/h\|_1 \to 0$. Since $hF = \bar{u}kF = \bar{u}\frac{F}{|F|+1} \in L^\infty$, we can conclude that

$$\|h_n hF - F\|_1 = \|h_n hF - \frac{1}{h}hF\|_1 \leq \left\|h_n - \frac{1}{h}\right\|_1 \|hF\|_\infty \to 0.$$

For each $n \in \mathbb{N}$ and every $g \in M$, from Lemma 3.5 we know that $h_n hg \in H^\infty M \subset M$. Hence

$$\int_{\mathbb{T}} gh_n hFdm = \int_{\mathbb{T}} h_n hgFdm = 0, \qquad \forall g \in M.$$

Suppose $g \in [M]^{-\alpha}$. Then there is a sequence $\{g_m\}$ in M such that $\alpha(g_m - g) \to 0$ as $m \to \infty$. For each $n \in \mathbb{N}$, it follows from $h_n hF \in H^\infty L^\infty \subset L^\infty$ that

$$\begin{aligned} \left| \int_{\mathbb{T}} g h_n h F dm - \int_{\mathbb{T}} g_m h_n h F dm \right| &\leq \int_{\mathbb{T}} |(g - g_m) h_n h F| dm \\ &\leq \|h_n h F\|_\infty \int_{\mathbb{T}} |g_m - g| dm \\ &= \|h_n h F\|_\infty \|g_m - g\|_1 \\ &\leq \|h_n h F\|_\infty \alpha(g_m - g) \to 0. \qquad (m \to \infty) \end{aligned}$$

Thus

$$\int_{\mathbb{T}} g h_n h F dm = \lim_{m \to \infty} \int_{\mathbb{T}} g_m h_n h F dm = 0, \qquad \forall g \in [M]^{-\alpha}.$$

In particular, $w \in [M]^{-\alpha} \cap L^\infty$ implies that $\int_{\mathbb{T}} h_n h F w dm = \int_{\mathbb{T}} w h_n h F dm = 0$. Hence,

$$\begin{aligned} 0 &\neq \left| \int_{\mathbb{T}} F w dm \right| \\ &= \lim_{n \to \infty} \left| \int_{\mathbb{T}} F w dm \right| \\ &\leq \lim_{n \to \infty} \left| \int_{\mathbb{T}} F w - h_n h F w dm \right| + \lim_{n \to \infty} \left| \int_{\mathbb{T}} h_n h F w dm \right| \\ &\leq \lim_{n \to \infty} \|F - h_n h F\|_1 \|w\|_\infty + 0 \\ &= 0, \end{aligned}$$

a contradiction. Hence $M = [M]^{-\alpha} \cap L^\infty$.

2. Let $M = W \cap L^\infty$. To prove M is weak*-closed in L^∞, using the Krein-Smulian theorem in Lemma 3.6, we need only show that $M \cap \mathbb{B}$ is weak*-closed. It is clear that $M \cap \mathbb{B} = W \cap \mathbb{B}$. From part (2) of Lemma 2.9, $M \cap \mathbb{B} = W \cap \mathbb{B}$ is α-closed. Since α is continuous, it follows from part (1) of Lemma 2.9 that $M \cap \mathbb{B}$ is $\|\|_2$-closed. The fact that $M \cap \mathbb{B}$ is convex implies M is closed in the weak topology on L^2. If $\{f_\lambda\}$ is a net in $\mathbb{B}$ and $f_\lambda \to f$ weak* in L^∞, then, for every $g \in L^1$, $\int_T (f_\lambda - f)\, g dm \to 0$. But $L^2 \subset L^1$, so $f_\lambda \to f$ weakly in L^2. Hence $M \cap \mathbb{B}$ is weak*-closed in L^∞.

3. Since W is α-closed in L^α, it is clear that $W \supset [W \cap L^\infty]^{-\alpha}$. Suppose $f \in W$ and let $k = \frac{1}{|f|+1}$. Then $k \in L^\infty$ and $k^{-1} \in L^\alpha$. It follows from Lemma 3.3 that there is an $h \in H^\infty$, $1/h \in H^\alpha$ and an unimodular function u such that $k = uh$, so $hf = \bar{u} k f = \bar{u} \frac{f}{|f|+1} \in L^\infty$. There is a sequence $\{h_n\}$ in H^∞ such that $\alpha\left(h_n - \frac{1}{h}\right) \to 0$. For each $n \in \mathbb{N}$, it follows from Lemma 3.5 that $h_n h f \in H^\infty H^\infty W \subset W$ and $h_n h f \in H^\infty L^\infty \subset L^\infty$, which implies that $\{h_n h f\}$ is a sequence in $W \cap L^\infty$. From part (2) of Lemma 2.2,

$$\alpha\left(h_n h f - f\right) \leq \alpha\left(h_n - \frac{1}{h}\right) \|hf\|_\infty \to 0.$$

Thus $f \in [W \cap L^\infty]^{-\alpha}$. Therefore $W = [W \cap L^\infty]^{-\alpha}$. $\square$

A quick corollary of the preceding result is the following conclusion, which gives us a quick route from the $\|\|_2$-version of invariant subspace structure to the (weak*-closed) $\|\|_\infty$-version of invariant subspace structure.

Corollary 3.8. *A weak*-closed linear subspace M of L^∞ satisfies $zM \subset M$ if and only if $M = \phi H^\infty$ for some unimodular function φ or $M = \chi_E L^\infty$ for some Borel subset E of $\mathbb{T}$.*

Proof. Since $zM \subset M$, it is easy to check that $[M]^{-\|\|_2}$ satisfies $z[M]^{-\|\|_2} \subset [M]^{-\|\|_2}$. Hence the Beurling-Helson-Lowdenslager theorem for $\|\|_2$ (Theorem 1.1) implies $[M]^{-\|\|_2} = \varphi H^2$ for some unimodular function φ or $[M]^{-\|\|_2} = \chi_E L^2$ for some Borel subset E of $\mathbb{T}$. It follows from part (1) of Theorem 3.7 that $M = [M]^{-\|\|_2} \cap L^\infty$ equals $\varphi H^2 \cap L^\infty = \varphi H^\infty$ or $\chi_E L^2 \cap L^\infty = \chi_E L^\infty$. $\square$

The following theorem is the generalized Beurling-Helson-Lowdenslager theorem for a continuous $\|\|_1$-dominating normalized gauge norm.

Theorem 3.9. *Suppose α is a continuous $\|\|_1$-dominating normalized gauge norm and W is a closed subspace of L^α. Then $zW \subseteq W$ if and only if either $W = \phi H^\alpha$ for some unimodular function ϕ or $W = \chi_E L^\alpha$ for some Borel set $E \subset \mathbb{T}$. If $0 \neq W \subset H^\alpha$, then $W = \varphi H^\alpha$ for some inner function φ.*

Proof. The "only if" part is obvious. Let $M = W \cap L^\infty$. It follows from part (2) of Theorem 3.7 that M is weak*-closed in L^∞. Since $zW \subset W$, it is easy to check that $zM \subset M$. Then by Corollary 3.8, we can conclude that $M = \phi H^\infty$ for some unimodular function φ or $M = \chi_E L^\infty$ for some Borel subset E of $\mathbb{T}$. It follows from part (3) of Theorem 3.7 that $W = [W \cap L^\infty]^{-\alpha} = [M]^{-\alpha}$, so $W = [\varphi H^\infty]^{-\alpha} = \varphi H^\alpha$ for some unimodular function ϕ or $W = [\chi_E L^\infty]^{-\alpha} = \chi_E L^\alpha$ for some inner function φ. If $0 \neq W \subset H^\alpha$, we must have $W = \varphi H^\alpha$, so the unimodular function $\varphi = \varphi \cdot 1 \in \varphi H^\alpha = W \subset H^\alpha$, which implies φ is an inner function. $\square$

Remark 3.10. *In Theorem 3.9, there are two situations of the invariant subspace structure as follows:*

(1) *If $W = \chi_E L^\alpha$, then $zW = W$, which means that W is a doubly invariant subspace in L^α.*
(2) *If $W = \varphi H^\alpha$, then $zW \subsetneqq W$. In fact, since the multiplication operator M_φ is an isometry on L^α and $zH^\alpha \subsetneqq H^\alpha$, we see $\varphi z H^\alpha \subsetneqq \varphi H^\alpha$, which means $zW = \varphi z H^\alpha \subsetneqq \varphi H^\alpha = W$. This means that W is a simply invariant subspace in L^α.*

4. Acknowledgements

The author is grateful to her advisor, Don Hadwin, and to Professor Eric Nordgren, who introduced her to Hardy spaces. They both provided help and support.

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